Game theory applied to Protection against SIS Epidemics in Networks

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CONGAS

• European project on games in complex systems:
  • Dynamics and COevolution in Multi Level Strategic iNteraction GAmes

• Objectives of talk: give insight for the mathematical tools, models and analysis.
Definition of the states in SIS

- Each node $j$ can be in either of the two states:
  - “0”: healthy
  - “1”: infected
- **Markov continuous time:**
  - infection rate $\beta$
  - curing rate $\delta$
- Mathematically:
  - $X_j$ is the state of node $j$
  - infinitesimal generator $Q_j(t) = \begin{bmatrix} -q_{1,j} & q_{1,j} \\ q_{2,j} & -q_{2,j} \end{bmatrix} = \begin{bmatrix} -q_{1,j} & q_{1,j} \\ \delta & -\delta \end{bmatrix}$
SIS model

- Using the standard $N$-Intertwined mean-field approximation (NIMFA), the SIS governing equation for node $i$ is:

$$\frac{d\nu_i(N; t)}{dt^*} = -\nu_i(N; t) + \tau(1 - \nu_i(N; t)) \sum_{j=1}^{N} a_{ij} \nu_j(N; t),$$

with $t^* = \delta t$ and $\tau = \frac{\beta}{\delta}$ is the effective spreading rate.

- The steady-state equations, valid for any graph are

$$\nu_i(\infty)(N) = 1 - \frac{1}{1 + \tau \sum_{j=1}^{N} a_{ij} \nu_j(\infty)(N)}.$$

- These steady-state equations only have two possible solutions: the trivial $\nu_i(\infty)(N) = 0$, corresponding to the exact absorbing state in SIS epidemics, and the non-trivial solution, corresponding to the metastable SIS regime.
Quasi stationary regime

- No external source of viruses
- Therefore 0 is absorbing
- But in the epidemic regime the system stays away long time away from 0
- Has time to reach a quasi stationary regime
- Various definitions of quasi stationarity:
  1. Stationary regime of a MC obtained by replacing the transition probab by those conditioned on not reaching 0
  2. Consider the limit for system with constant arrival of viruses as their arrival rate goes to zero
NIMFA: N-intertwined mean-field approxim.

\[
\frac{dE[X_j]}{dt} = -\delta E[X_j] + \beta \sum_{k=1}^{N} a_{kj} E[X_k] - \beta \sum_{k=1}^{N} a_{kj} E[X_jX_k]
\]

\[E[X_jX_k] = \Pr[X_j = 1, X_k = 1] = \Pr[X_j = 1|X_k = 1] \Pr[X_k = 1]\text{ and } \Pr[X_j = 1|X_k = 1] \geq \Pr[X_j = 1]\]

\[E[X_iX_k] \geq \Pr[X_i = 1] \Pr[X_k = 1] = E[X_i] E[X_k]\]

\[
\frac{dE[X_j]}{dt} \leq -\delta E[X_j] + \beta \sum_{k=1}^{N} a_{kj} E[X_k] - \beta E[X_j] \sum_{k=1}^{N} a_{kj} E[X_k]
\]

NIMFA (= equality above) upper bounds the prob. of infection

Scaling

• Define a k-scaled network to be one where each node is replaced by a cluster of k nodes s.t. if i is a neighbour of j in the original net, then all nodes in the corresponding clusters are neighbours in the scaled net.

• Then the infection probab are the same in each cluster

• Assume that tau is k times smaller in the scaled net
Insight on NIMFA

• Works better for large populations
Invariance property

• It is seen that the infection probability does not depend on the scale parameter $k$. Indeed

\[
\frac{dv_i(N; t)}{dt^*} = -v_i(N; t) + \tau(1 - v_i(N; t)) \sum_{j=1}^{N} a_{ij} v_j(N; t),
\]

• But MF theorems show that only for large $k$ the scale system is a good approximation.
• NIMFA thus obtains the infection probability independently of $k$ but it is a good approximation only for large $k$
Graph topologies

- Fully connected
- Bipartite graph
- Multi community
- Extensions
Examples

For a full mesh network, we have:

\[ v_{i\infty}(N) = \begin{cases} 
1 - \frac{1}{\tau(N-1)}, & \text{if } \tau \geq \frac{1}{N-1}, \\
0, & \text{otherwise.} 
\end{cases} \]

For a bipartite network with \( M \) and \( N \) nodes, we have:

\[ v_m^m = \frac{\tau^2 MN - 1}{\tau M(\tau N + 1)} \quad \text{and} \quad v_n^n = \frac{\tau^2 MN - 1}{\tau N(\tau M + 1)}. \]

For a multi-communities network with \( M \) groups, we have:

\[ v_m(N_m, v_\infty) = 1 - \frac{1}{1 + \tau_m(N_m - 1)v_m + \tau_m v_\infty}, \]

with \( v_\infty \) is the infection probability for the core node.
Protection non-zero sum game

• Each node can decide whether or not to invest in vaccination.
• Vaccination cost per node: \( S_{1,n} = C \)
• Expected healing cost: \( S_{0,n} = H \nu_{i,\infty}(n) \)
• Pure equilibrium is identified with some integer number \( n^* \) that do not invest such that no one gain by deviating.
• Note that it is not symmetric in general
Equilibrium. 2 conditions

• If at equilibrium node k invests then he cannot gain more by not investing hence $S_{0,n^*+1} \geq C$

• *Note:* $S_{0,n}$ increases in n this implies that even more than one simultaneous deviation is not beneficial

• If a node does not invest then it cannot gain more by investing $S_{0,n^*} \leq C$

• This is also the condition for more than one simultaneous deviation.

• Hence a Nash equilibrium is a strong equilibrium
Existence, uniqueness of equilibrium

• Since $S_{0,n^*}$ is strictly increasing in $n^*$, there is at most one $n^*$ that satisfies the equilibrium condition. Therefore we have uniqueness

• If the condition is not satisfied then the equilibrium consists of all players investing or all players non investing. Hence existence.
Congestion games (Rosenthal)

- Consider a network $G=(K,V)$, $N$ players each with a given source $(S)$ destination $(D)$ pair.
- Each player has to route a single unit through a path (set of consecutive links between its source and its destination). No splitting!
- The cost of each link is an increasing function of the number of flows sent through it. Path cost is the sum of costs of the links along the path.
- An equilibrium is a set of paths for the players such that no player can decrease its cost by deviating unilaterally.
Equivalent game

- The value of \( a(i) \) that maximises the utility function \( J(i;a) \) of player \( i \) is unchanged if we add to \( J(i;a) \) a constant.

- It is unchanged if we add to \( J(i;a) \) a function of the actions of players \( j \neq i \).

- It is also unchanged if we replace it with \( h(J(i;a)) \) where \( h \) is any strictly monotone increasing function.

Example: the power control game

- Let \( |p| := \sum_{j=1}^{I} p(j) \). Then for given \( p(j), j \neq i, \)

\[ p(i) \text{ maximises } Thp(i) + g \cdot p(i) \text{ if and only if it maximises } Thp(i) + g \cdot |p| \]
Potential

\[ Thp(i) = \log \left( 1 + \frac{p(i)}{N + \sum_{j \neq i} p(j)} \right) \]

\[ = \log \left( N + \sum_{j=1}^{I} p(j) \right) - \log \left( N + \sum_{j \neq i} p(j) \right) \]

Note that the second term does not depend on \( p(i) \). Thus \( p(i) \) maximises \( J(i;p) \) if and only if it maximises

\[ V(|p|) := \log \left( N + |p| \right) - g |p| \]

\( V \) is the same for all players. Any optimal solution of \( \max V \) is an equilibrium for the original game. \( V \) is called a potential.
Potential Games

- $V$ is a potential if for every $i$ and every vector $a$ and action $b(i)$

$$J(i; a) - J(i; [a(-i), b(i)]) = V(a) - V([a(-i), b(i)])$$

Equivalently, for all $i$

$$\frac{\partial V(a)}{\partial a(i)} = \frac{\partial J(i; a)}{\partial a(i)}$$
Convergence to equilibrium

- Define a best response sequence as a couple $(t(n), i(n))$ where $t(n)$ is a strictly increasing sequence of times such that at time $t(n)$ player $i(n)$ updates its action using a best response to the current actions of the other players. We assume that each player appears infinitely often in $i(n)$.

- A change of a policy by a player results in higher utility if and only if it results in a higher potential.

- This implies that any sequence of best responses of the players converges to a local maximum of the potential in finite time.

- If the potential is strictly concave then it has a unique local maximum which is a global maximum.

- Therefore there is a unique equilibrium and any sequence of best responses of the players converges to it in finite time.
Congestion games over parallel links [Rosenthal]

• There are $M$ parallel links between a source $S$ and a destination $D$.
• Each of $I$ players have one unit flow to ship. It has to decide through which link to ship that flow.
• Let $x(m)$ be the total flow sent to link $m$ under some action vector $a$
• The cost to ship a unit of flow through link $m$ is $f(m, x(m))$
• Assume player $i$ decides to change from link $k$ to $n$. Then
  \[ C(i; a) - C(i ; [ a(-i), n ] ) = f(k, x(k)) - f(k, x(k)-1) - [ f(n, x(n) + 1) - f(n, x(n)) ] \]
• Define
  \[ V(x) = \sum_{m=1}^{M} v(m, x) \text{ where } v(m, x) := \sum_{j=1}^{x(m)} f(m, j) \]
Then $V$ is a potential.
Other equilibria

• Symmetric mixed equilibrium $p$ exists, obtained by the indifference principle: $p$ is such that if $N-1$ players play $p$ then player $N$ is indifferent to invest or not.

• Uniqueness follows since from monotonicity: if under $p$ a player is indifferent then the player cannot be indifferent for $p' > p$ since the noninvestment cost under $p'$ is larger than under $p$ and therefore player $N$ is not indifferent. Thus $p'$ cannot be best response.
Extension: multiclass networks

• Consider K classes. Assume that being infected is perceived differently in different classes: H and C are class dependent.

• The game is no more a congestion game and it does not have anymore a potential.

• Such games are called CROWDING GAMES [Milchtaich]. They still have a pure equilibrium, but convergence is not guaranteed anymore.
Multi community topology

• Let the infection quasi stable probability of the core node be given by $u$.
• Focus on community $k$. Given $u$, there is no dependence on other communities. We add to the balance equations of nodes in the community a term for infection from the core node.
• We can describe the protection decisions of nodes in com $k$ as a congestion game.
Multi community topology

• The pure equilibrium $n(k,u)$ in each com $k$ depends on $u$. So does the infection probability vector $V(k,u)$ $(k=1,...,K)$ for nodes at the community $k$.

• On the other hand, a fixed vector $V(k)$ of infection probability in the coms determines $u$.

• We obtain the equilibrium using a fixed point argument.
We can extend to more than 2 states

- An infection may have several states. EG the cost may increase in the infection state.
- More complex healing procedures. It may be less expensive to rejuvenate an infected node than to heal it completely.
- We may distinguish between contagious and infected nodes. As long as symptoms of the infection do not appear, a node is contagious, unless a special effort (sampling) is made to determine node's state.
- If there is noise then we may have only estimates of the state, and we may compute policies that achieve constraints on false positive or false negatives
Hierarchical game

• For fixed rejuvination and sampling policies (and costs) we get again a congestion game.
• We can optimize the performance at equilibrium by properly choosing rejuvination, healing and sampling policies.
• Collaboration with Siemens
Extension: net formation, connectivity constraints

• N nodes. A constant price for each link formed. Plus a cost proportional to the infection probability
• [Rosen type] coupled constraints: the network has to be connected.
• At equilibrium, a deviation of a node results either in disconnectivity or in higher cost.
• Why should node j care if nodes r and s are connected?
• Theorem: there exists a pure equilibrium. All equilibria are trees, not vice versa.
• Examples: start topology, and a line topology.
• Equilibrium selection issues: for some values of tau, the best and worst equilibrium are neither star nor line.
References

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References

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