Maximizing Influence in Competitive Environments: A Game-Theoretic Approach

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Outline

- Competing ideas in social networks
- Relevant properties of submodular functions and connected components
- Model for Dynamic Influence in Competitive Environments (DICE)
- Influence maximization games using DICE
  - Leader-follower game
  - Simultaneous-move game
- Simulation results with Wiki-vote

Competing Ideas in Social Networks

- Ideas spread through social networks
  - Word-of-mouth, blogs, online social networks,...
- Multiple ideas may propagate through same network

Rival advertisements

Political campaigns

Rumor spreading

- Possible scenarios
  - One idea introduced first, followed by others
  - All ideas introduced simultaneously
  - Reinforcing ideas introduced over time **(not considered today)**

Modeling Spread of Ideas

- Spreading ideas lead to following questions:
  - How does a set of individuals adopting an idea changes over time?
  - What fraction of population will eventually adopt each idea?
  - Who should be initially seeded with an idea to maximize its spread?
- **Our Contribution:** A tractable analytical model for answering the above questions within a game-theoretic framework for two ideas

Background: Submodular Functions

- Let V denote a finite set, and let $f : 2^V \to \mathbf{R}$ map subsets of V to real numbers
- Function f is submodular if for any S ⊆ T ⊆ V and v ∉ T,

$$f(S \cup \{v\}) - f(S) \geq f(T \cup \{v\}) - f(T)$$

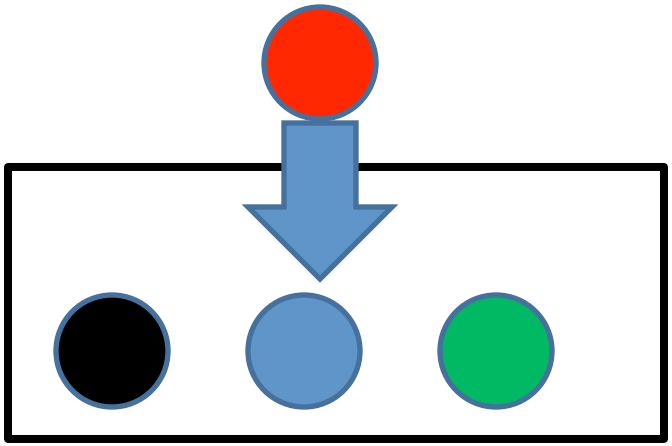
- Intuitively, this is a *diminishing returns* property
- Properties of submodular functions:
  - Nonnegative weighted sum of submodular functions is submodular:

$$\begin{aligned}(\alpha f_1 + \beta f_2)(S \cup \{v\}) - (\alpha f_1 + \beta f_2)(S) &= \alpha f_1(S \cup \{v\}) - \alpha f_1(S) + \beta f_2(S \cup \{v\}) - \beta f_2(S) \\ &\geq \alpha(f_1(T \cup \{v\}) - f_1(T)) + \beta(f_2(T \cup \{v\}) - f_2(T)) \\ &= (\alpha f_1 + \beta f_2)(T \cup \{v\}) - (\alpha f_1 + \beta f_2)(T)\end{aligned}$$

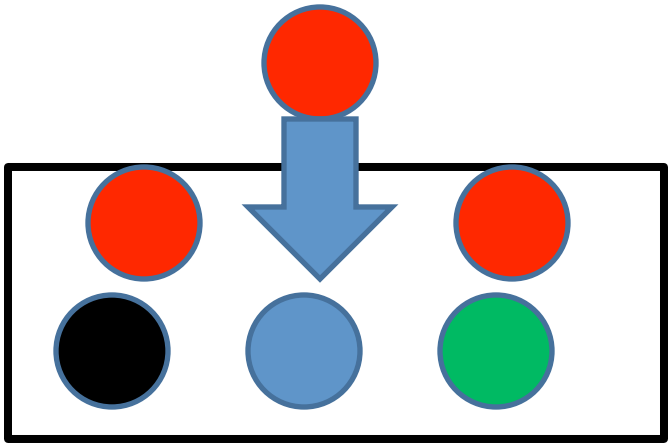
  - **If f is submodular**, then choosing a set S with |S| = k that maximizes f(S) can be approximated in polynomial time up to a bound of (1-1/e)

Examples of Submodular Functions

- **First example:** Let S represent a set of colored balls, and let f(S) represent the number of colors in S



Number of colors increased by one



No effect on number of colors

- **Second example :** Let S be a set of vectors, and let f(S) be the rank of S

  - Adding a vector to S increases the rank by one, unless it is in the span of S

Background: Connected Component

- A directed graph (digraph) is strongly connected if there is a path between any two vertices with correct edge orientations
- A strongly connected component is a maximal strongly connected subgraph of a digraph

Propagation of Ideas: Related Work

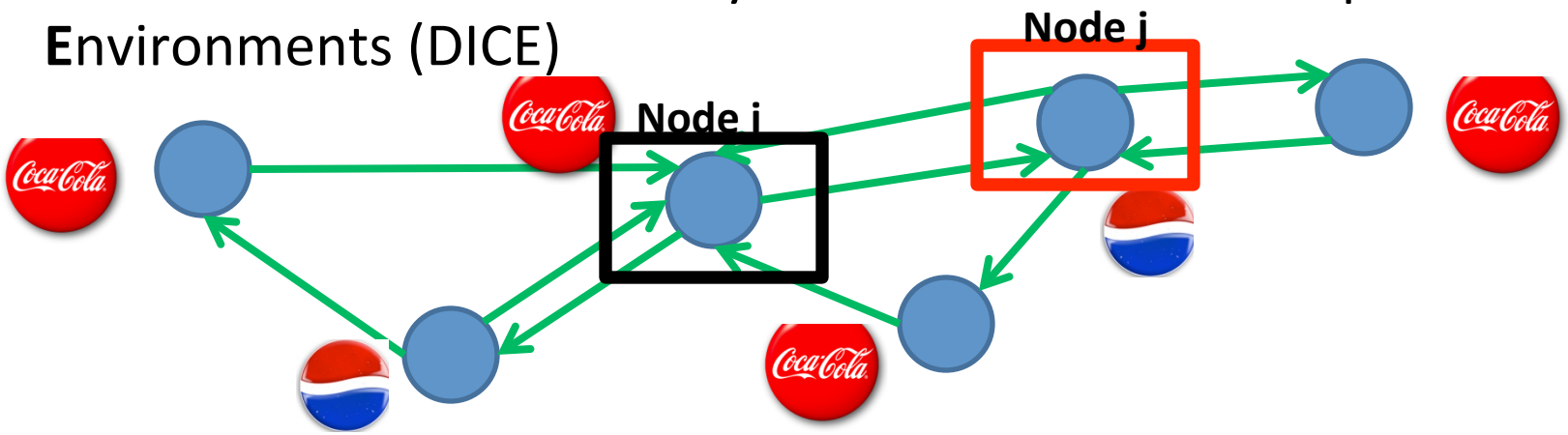
- Threshold model [Granovetter 1973]
- Cascade model [Goldenberg 2001]
- Generalized cascade model [Kempe 2003]
- Markovian models [Boyd 2005, Acemoglu 2010]

Maximizing Influence: Related Work

- Maximizing spread of a single idea
  - Goal: Choose a set S of individuals to initially adopt and propagate the idea
  - Number of individuals eventually holding idea is submodular function of S under generalized cascade model [Kempe 2003]
- Propagation of competing ideas [Bharathi 2007, Kostka 2008]
  - Extends generalized cascade model to multiple ideas
  - Individuals cannot switch between ideas
  - Formulate leader-follower games in which one player chooses a set S_1 and other player chooses a set S_2 after observing S_1
  - Reduces to facility location games, which are NP-hard to approximate

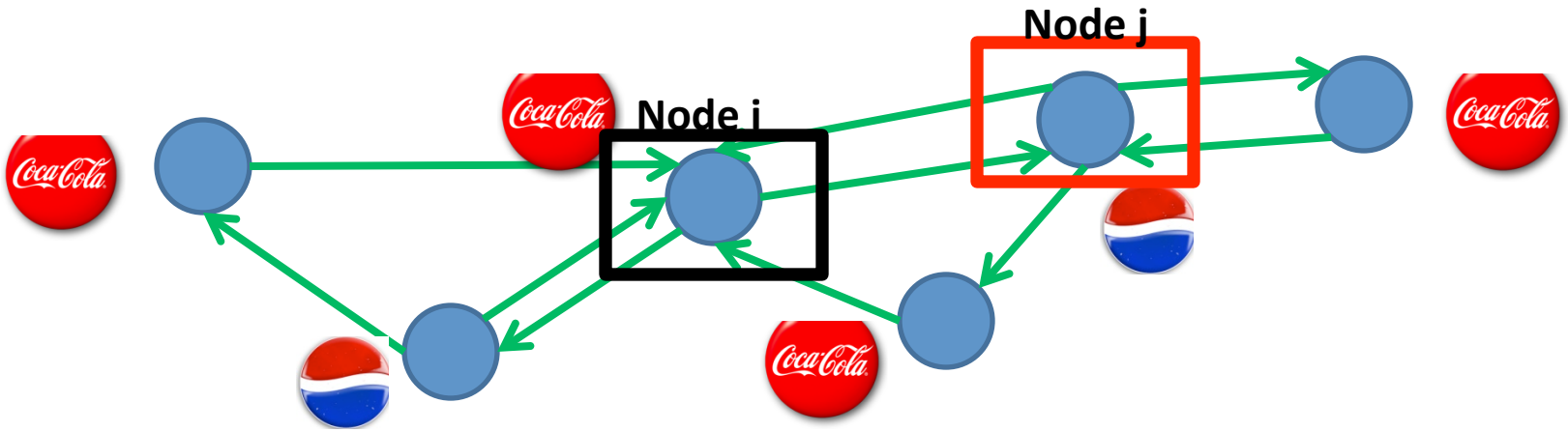
Our Competitive Influence Propagation Model

- We introduce a model of **D**ynamic **I**nfluence in **C**ompetitive **E**nvironments (DICE)



- Set of individuals (nodes) V = {1,...,n}, set of social relationships E (represented by links), set of ideas ***I***= {I_1, ..., I_m}
- At each time t, each individual i interacts with one of its neighbors
  - Interacts with j with probability d_{ij}
  - i.e., i meets j in person or reads j's blog
- With probability d_{ii}, i does not meet any neighbor; $d_{ii} = 1 - \sum_{j \in N(i)} d_{ij}$

Dynamic Influence in Competitive Environments (DICE)



- Set of individuals (nodes) V = {1,...,n}, set of social relationships E (represented by links), set of ideas ***I***= $\{I_1, ..., I_m\}$
- If i has adopted idea I_k and j has adopted I_r:
  - Then i switches to idea I_r with probability $P_{ij}(k,r)$
  - Otherwise, i adds I_r to a list of known ideas that i may adopt later
- If node i does not meet any neighbor at time t, i independently switches to a different known idea (i.e. I_s) with probability $P_i(k,s)$

Steady-state Distribution of Ideas

- **Question:** How many nodes will adopt each idea asymptotically?
- Define random variable $X_i(t)$ = idea held by node i at time t; let $X(t) = (X_1(t), \ldots, X_n(t))$
- By definition of DICE, $X = \{X(t): 0 \leq t < \infty\}$ forms a Markov chain
- If stationary distribution exists, then it gives asymptotic behavior of X

Existence of Stationary Distribution

- Define V_k = set of nodes adopting idea I_k at t=0
- The stationary distribution of X exists and depends only on $V_1,\ldots,V_m$ if:
  - The probability $d_{ii} > 0$ for each i in V (every node avoids its neighbors with nonzero probability)
  - The probability $P_i(k,l) > 0$ for every i, k, and l (every node changes its mind with some probability)
- **Irreducibility**: $d_{ii} > 0$ and $P_i(k,l) > 0$ means nonzero probability of transitioning between any two states at each step
- **Aperiodicity**: $d_{ii} > 0$ and $P_i(k,k) > 0$ implies that state remains unchanged at each step with nonzero probability

Utilities Based on Stationary Distribution

- Define $\pi_i(k|V_1,\ldots,V_m)$ to be the probability that node i adopts idea I_k in the stationary distribution
- The utility of the owner of I_k is then the average number of individuals holding I_k in steady-state, given as

$$U_k(V_1,\ldots,V_m) = \sum_{i \in V} \pi_i(k|V_1,\ldots,V_m)$$

- Uncertainty in social relationships E, represented as a distribution P(E) over subsets of E_0, leads to average utility

$$U_k(V_1,\ldots,V_m) = \sum_{E \subseteq E_0} \sum_{i \in V} \mathcal{P}(E)\pi_i(k|V_1,\ldots,V_m,E)$$

- How should the owners of $I_1,\ldots, I_m$ select $V_1,\ldots,V_m$ in order to maximize their utilities?

Leader-Follower Influence Game

- Two players, each with one idea (m=2)
- One player (leader) introduces idea first by choosing V_1
- Second player (follower) observes V_1 and chooses V_2
- Adding a node to V_1 or V_2 incurs a cost c
- Leader's utility function given by:

$$U_1(V_1, V_2) = \sum_{E \subseteq E_0} \sum_{i \in V} \mathcal{P}(E) \pi_i(1|V_1, V_2, E) - c|V_1|$$

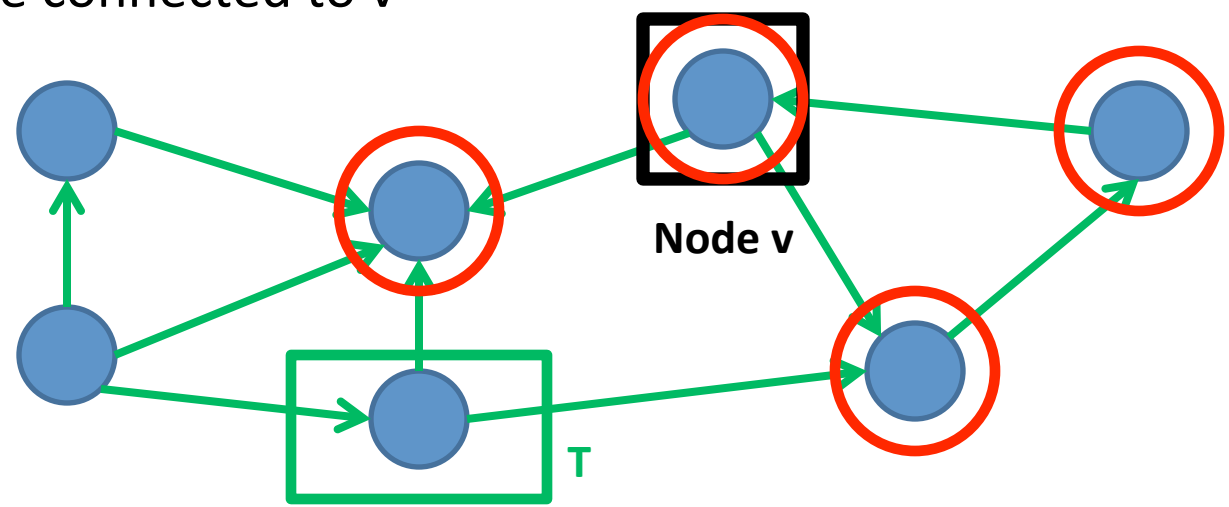
- Follower's utility function:

$$U_2(V_1, V_2) = \sum_{E \subseteq E_0} \sum_{i \in V} \mathcal{P}(E) \pi_i(2|V_1, V_2, E) - c|V_2|$$

- What are the optimal strategies for each player?

Maximizing Follower Utility

- Follower's goal: Given V_1, choose V_2 to maximize $U_2(V_1,V_2)$
- Observation: Stationary distribution $\pi_i(k|V_1,V_2)$ depends only on ideas known to i in steady state
- Node i will know idea I_k **if and only if** it is connected to a node in V_k
- Hence benefit of adding node v to V_2 depends only on which nodes are connected to v



When V_2 is initially empty, gain from adding node v to V_2 is 4

When V_2 is initially equal to T, gain from adding node v to V_2 is 0

- This implies that $U_2(V_1, V_2)$ is submodular as a function of V_2

Maximizing Follower Utility – Derivation

- Follower's goal: Given V_1, choose V_2 to maximize $U_2(V_1,V_2)$
- Observation: Stationary distribution $\pi_i(k|V_1,V_2)$ depends only on ideas known to i in steady state
- This depends on which connected components intersect V_2
- Benefit of adding a node v from component G_l to V_2 is therefore given by

$$U_2(V_1, V_2 + v) - U_2(V_1, V_2) = \begin{cases} \sum_{i \in G_l} \pi_i(2|V_1, v) - c, & V_2 \cap G_l = \emptyset \\ -c, & \text{else} \end{cases}$$

- Benefit is positive iff $V_2 \cap G_l \neq \phi$. Since $T \cap G_l \neq \phi$ implies $S \cap G_l \neq \phi$ when $S \subseteq T$, U_2 satisfies diminishing returns property and hence is submodular.
- In case of probabilistic topology, U_2 is nonnegative weighted sum of submodular functions, therefore still submodular

Solution Algorithms for Follower

- $U_2(V_1,V_2)$ can be maximized using a greedy approach
- Initialize $V_2 = \phi$
- At t-th iteration, add node v that maximizes

  $U_2(V_1, V_2 \cup \{v\}) - U_2(V_1,V_2)$
- Algorithm terminates when $U_2(V_1, V_2 \cup \{v\}) \leq U_2(V_1,V_2)$ for all v
- **Theorem:** The greedy algorithm returns a set $\tilde{V}_2$ such that

$$U_2(V_1, V_2^*) - U_2(V_1, \tilde{V}_2) \leq c|V_2^*|$$

Where V_2^* is the optimum selection for the follower

- Proof: Based on [Nemhauser 1978], Proposition 4.1

Maximizing the Leader's Utility

- Leader chooses set of nodes V_1 based on follower's best response, $V_2^*(V_1)$
- Let $\pi_i(2|I_1,I_2)$ denote the probability that i holds I_2 in steady state, given that both I_1 and I_2 are known to i
- Let G_l represent a strongly connected component of V
- For a given topology, the follower will select one node from G_l and add it to the set V_2 if the benefit outweighs the cost:

$$\sum_{i \in G_l} \pi_i(2|I_1, I_2) > c.$$

- Hence the leader's utility is given by

$$\sum_{l:G_l \cap V_1 \neq \emptyset} \sum_{i \in G_l} w_i - |V_1|c$$

where $w_i = \begin{cases} \pi_i(1|I_1, I_2), & \sum_{i \in G_l} \pi_i(2|I_1, I_2) > c \\ \pi_i(1|I_1) & else \end{cases}$

Solution Algorithms for Leader

- The incremental benefit from adding a node v from strongly connected component G_l to leader set V_1 is then given by

$$U_1(V_1 + v, V_2^*(V_1 + v)) - U_1(V_1, V_2^*(V_1)) = \begin{cases} \sum_{i \in G_l} w_i - c, & V_1 \cap G_l = \emptyset \\ -c, & \text{else} \end{cases}$$

- If $S \subseteq T$, then $T \cap G_l = \phi$ implies $S \cap G_l = \phi$
- Hence incremental benefit from adding v to T is no more than incremental benefit from adding v to S
- $U_1(V_1, V_2^*(V_1))$ is therefore a submodular function of V_1, can be solved by algorithm analogous to follower solution algorithm

Simultaneous Influence Game

- Occurs when two ideas introduced at the same time
- Utility functions are the same, but V_2 chosen without observing V_1
- **Goal:** Find strategies such that neither player has incentive to deviate unilaterally (Nash equilibrium)

$$V_1^* = \arg\max_{V_1} U_1(V_1, V_2^*)$$

$$V_2^* = \arg\max_{V_2} U_2(V_1^*, V_2)$$

Finding Nash Equilibria

- Consider a network which is deterministic (no uncertainty) and strongly connected
- Strategies reduce to: introduce idea or not?
- Let $B_k = \sum_{i \in G} \pi_i(k)$ and define H and H' to be strategies of introducing and not introducing idea, respectively
- Game described by payoff matrix

	H	H'
H	$(B_1 - c, B_2 - c)$	$(n - c, 0)$
H'	$(0, n - c)$	$(0, 0)$

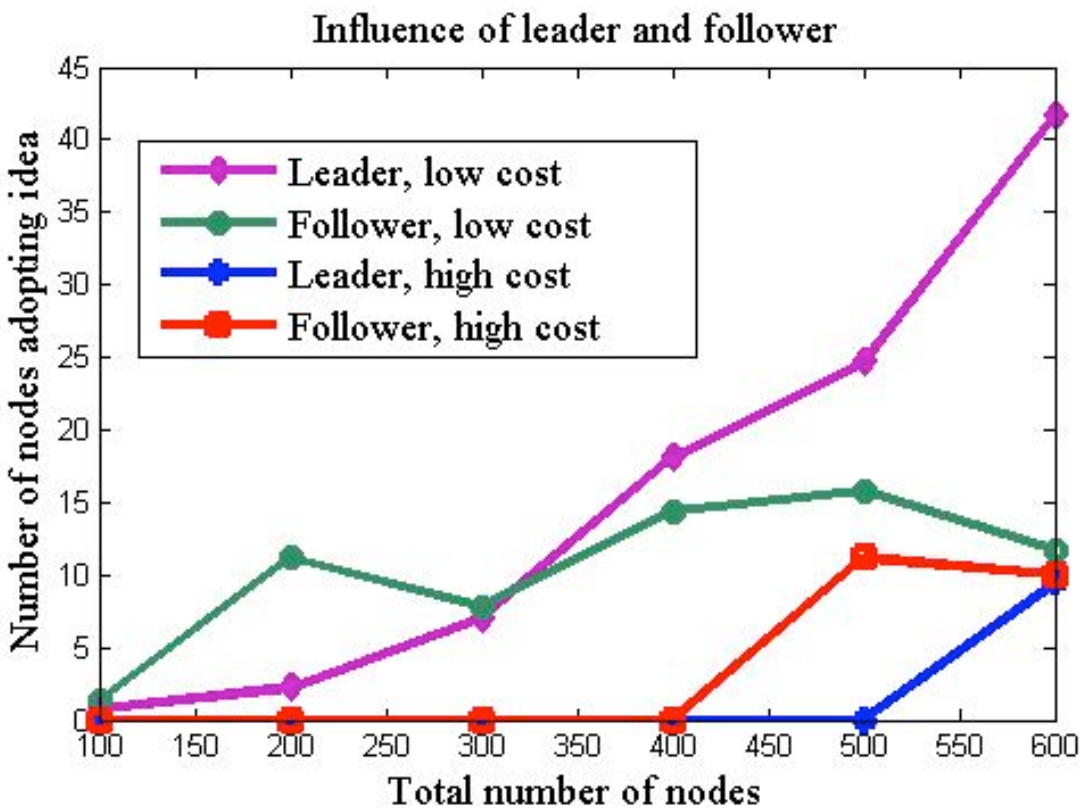
- If $B_k > c$ for k=1,2, then (H,H) is Nash equilibrium
- If c > n, then (H',H') is Nash equilibrium
- $B_k < c < n$, then two pure-strategy equilibria and one mixed strategy equilibrium

Simulation Study

- Simulations performed using Wiki-vote social network dataset
- Nodes represent Wikipedia users, an edge (i,j) exists if user i has influenced user j
- Leader-follower game simulated
- Parameters chosen:
  - Number of ideas m = 2
  - d_{ii} = 0.5 for all i
  - d_{ij} = 0.5/degree(i) for all edges (i,j)
  - Probabilities $P_{ij}(k,l)$ chosen uniformly at random from [0,1]
  - Probabilities $P_i(k,l)$ chosen uniformly at random from [0,0.05]
  - Cost of choosing a seed node set to c=5 or c=15
  - Number of nodes n ranges from 100 to 700

Simulation Results

- Utility of each player increases with # nodes
- U1 > U2
  - Follower must decide whether to compete for each connected component
  - Leader can choose strategy so that follower has no incentive to compete
- As cost increases, each player may choose not to enter the market (barriers to entry)



Future Work

- Modeling reinforcing ideas
  - A new idea is introduced in order to strengthen or expand influence of existing idea
- Choosing nodes to adopt ideas after the initial time step
- Other utility functions
  - Transient dynamics of influence
  - Nonlinear cost of introducing idea

Questions?